

Characterization of multipartite entanglement

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In this paper, we provide a characterization of multipartite entanglement in terms of equivalence classes of states under local unitary transformation (LU) by demonstrating a simple method to construct all homogenous polynomials that are invariant under local unitary group (LUIPs) for any fixed degree. We give an upper bound on the degree of the LUIP such that the ring of LUIPs can be generated by LUIPs with degree lower than the bound. Our characterization can directly generate an algorithm whose output is a generating set of LUIPs. By employing the concept of LUIPs, we prove that multipartite entanglement is additive in the sense that two multipartite states are LU equivalent if and only if n -copies of these two states are LU equivalent for some n . The method for studying LU equivalence is also used to classify the different types of multipartite mixed entanglement according to equivalence classes of states under stochastic local operations and classical communication (SLOCC), where the pure states case was previously studied by Gour and Wallach using another approach.

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Introduction—Multipartite entanglement is considered as an essential asset to information processing and computational tasks, which include measurement-based quantum computation[1, 2], quantum error correction schemes[3], and quantum secret sharing[4]. The intriguing properties and potential applications of entanglement spark many literature dedicated to quantify it as a resource, and numerous results have been obtained. In bipartite case, it is well known that the vector of Schmidt coefficients, instead of any scalar, is a proper entanglement measure when exact transformations are considered. However, it is much more complicated for multipartite entanglement even in the asymptotic manner [5–7]. Even though great efforts and lots of progresses has been made, we are still far from completely understanding the non-local properties of multipartite entanglement.

Entanglement transformation can often be utilized to study the equivalence of entanglement. The following entanglement transformation problem is fundamental: whether two pure N -partite states $|\psi\rangle$ and $|\phi\rangle$ could be transformed into each other, assuming that each party may use only local operations on their respective systems with the help of unlimited two-way classical communication (LOCC). Theoretically, this LOCC equivalence class is defined such that within this class any two quantum states are interconvertible by local unitary (LU) operators [8]. This problem is significant for characterizing entanglement by noticing that LOCC could never increase entanglement, then two quantum states are LOCC(LU) equivalent indicates that their entanglement are exactly the same in any perspective of view. Therefore, multipartite entanglement are characterized according to their inter-convertibility under local unitary transformation. Often, one may release the local unitary operator restriction to invertible local transformation, the LOCC

equivalence then turns to the widely studied stochastic local operations and classical communication (SLOCC) equivalence[8–15].

In order to study the LU equivalent problem, the concept of the local polynomial invariants is presented in [16–18]. A complete classification has been obtained only for very few simple case, $2 \times 2 \times n$ system and $2 \times 2 \times 2 \times 2$ system[19–21]. Beyond that, and despite the extensive literature, very little is known, even for pure tripartite states, about the set of all such LU-invariant polynomials (LUIPs) except for few techniques that were used to construct some of the LUIPs for multi-partite states, see [22] as a very incomplete list. Very recently, another approach of studying the LU equivalence of n -qubit states is presented in [23]. Such method creatively deduces this problem into solving finite set of non linear equations whose variables are two-qubit unitaries. When it is unavailable to solve such equations, this method is no longer operational. Thus, a characterization of LUIPs for multipartite system is highly desirable.

In this paper, we give a characterization of multipartite entanglement by exploiting a systematic method to describe the ring of all LUIPs. In order to do so, we first employ some techniques and concepts of invariant theory to demonstrate an explicit upper bound of the degree of LUIPs for which any LUIP can be generated by LUIPs with degree less than the bound. We then provide an algorithm to construct all LUIPs with fixed degree. Our main idea here is to look at the ring of homogenous polynomials that are invariant under local unitary group applied on one fixed party, then the ring of LUIPs is the intersection of these rings, which can be computed by computing the intersection of finite dimensional subspaces. By using the concept of LUIPs, we are able to show that multipartite entanglement is additive in the sense that

two multipartite states are LU equivalent if and only if n -copies of these two states are LU equivalent for some n . Interestingly, this idea can be used to study the multipartite mixed entanglement in terms of equivalence classes of states under SLOCC. For pure state, we provide an algorithm to compute the whole set of SL-invariant polynomials (SLIPs) which was recently obtained by Gour and Wallach in [13] using another approach. The problem of the rank one SLOCC equivalence relation between mixed state is transformed into the problem of equivalence relation for pure states under the group of tensor of SL group and unitary group (SLU), then the whole set of SLU-invariant polynomials (SLUIPs) is constructed by employing our previous approach. It would be worth to notice that our algorithms are dramatically simple by only computing finite intersections between subspaces, which is of course feasible in all dimensions.

Preliminaries—In order to demonstrate our main result, we need to employ the result in [24] by Derksen: Let

group G over an algebraically closed field K of characteristic 0, acting on an s -dimensional vector space V as follows: there exist polynomials $h_1, \dots, h_\ell \in K[z_1, \dots, z_t]$ $a_{i,j} \in K[z_1, \dots, z_t]$ such that G is the zero set of these polynomials; On the other hand, there exist polynomials $a_{i,j}$ for $i, j \leq s$ such that $g : G \rightarrow \mathbb{GL}(V)$ is given by $g \rightarrow (a_{i,j}(g))_{i,j \leq s}$, where $\mathbb{GL}(V)$ is the general linear group of V , i.e., the group of invertible matrices.

The coordinate ring of V can be identified as $R = K[x_1, \dots, x_s]$, and G induces an action on R . The invariant ring of G on V , denoted as R^G , consists of those polynomials in R invariant under G , i.e.,

$$R^G = \{r : r(g \cdot v) = r(v), r \in R, \forall g \in G, v \in V.\}$$

It is known that R^G is finitely generated according to Hilbert's famous result [25]. The question here is to derive an explicit degree bound for this finite generation.

$$\begin{aligned} \beta(V, G) &= \min\{d \in \mathbb{N} : R^G \text{ is generated by invariants of degree } \leq d\}. \\ \sigma(V, G) &= \min\{d \in \mathbb{N} : \text{invariants of degree } \leq d \text{ defines the nullcone of } R^G\}. \end{aligned}$$

Let $A = \max\{\deg(a_{i,j}) \mid i, j \leq s\}$, $H = \max \deg(h_i)$, and $d = \dim(G)$ where $\dim(G)$ denotes the dimension of the algebraic variety G [26]. Derksen shows that [24]

$$\sigma(V, G) \leq H^{t-d} A^d, \text{ and } \beta(V, G) \leq \max(2, \frac{3}{8} \sigma^2).$$

Main Results—In this section, we demonstrate an explicit upper bound for rings of invariant, then provide an algorithm to compute the rings of invariant.

During this paper, we consider the Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n,$$

with d_i being the dimension of each Hilbert space \mathcal{H}_i .

Let $\mathbb{U}(d_j)$ be the group of $d_j \times d_j$ unitary group, local unitary group is defined as

$$\mathbb{LU} \equiv \mathbb{U}(d_1) \otimes \mathbb{U}(d_2) \otimes \dots \otimes \mathbb{U}(d_n).$$

For $|\Psi\rangle \in \mathcal{H}$, the orbit $\mathbb{LU}|\Psi\rangle := \{g|\Psi\rangle : g \in \mathbb{LU}\}$ is consisting of quantum states in the LU equivalent class of $|\Psi\rangle$. The entanglement of $|\Psi\rangle$ is the same as that of any $|\Phi\rangle \in \mathbb{LU}|\Psi\rangle$ from any point of view.

We are interested here characterizing multipartite entanglement by classifying classes of LU equivalent states. It is direct to see that the whole state space is classified into the union of distinct orbits. In order to separate distinct orbits, one may use a series of functions $f_m : \mathcal{H} \mapsto \mathbb{C}$ for $1 \leq m \in \mathbb{N}$. To maintain the definition is also well

defined on orbits, a natural requirement is that f_m is LU-invariants, i.e., invariant under local unitary transformations. As the simplest and the most well studied functions, f_m is chosen to be polynomial. Those polynomial which is invariant under local unitaries is called LUIP. More precisely, for the local unitary group, a function $f : \mathcal{H} \mapsto \mathbb{C}$ is an LUIP, if $f(|\Psi\rangle\langle\Psi|)$ is the homogeneous polynomial on entries of $|\Psi\rangle\langle\Psi|$ and

$$f(g|\Psi\rangle\langle\Psi|g^+) = f(|\Psi\rangle\langle\Psi|), \quad \forall g \in \mathbb{LU} \text{ and } \forall |\Psi\rangle \in \mathcal{H}.$$

It is known that the LUIPs can be used to determine whether two states in \mathcal{H} belong to the same LU class [18, 19], i.e., two pure state are LU equivalent if and only if $f_i(|\Psi\rangle\langle\Psi|) = f_i(|\Phi\rangle\langle\Phi|)$ holds for all LUIPs f_i . Since the degree of LUIP can be arbitrary large, it is not feasible to calculate $f_i(|\Psi\rangle\langle\Psi|)$ for all LUIP f_i .

Our first result is to show that the ring of LUIPs is finitely generated. Moreover, we demonstrate a degree bounds for a generator for the ring of LUIPs.

Theorem 1. *The set of all LUIPs is generated by the LUIPs with degree less than $N(d_1, d_2, \dots, d_n) = \frac{3}{8}(\prod_i d_i) \sum_i 2d_i^2 + 2$.*

Proof:—For a linear operation ρ in \mathcal{H} , $g \in \mathbb{LU} \leq \mathbb{GL}(\mathcal{H})$ acts on ρ by sending ρ to $g\rho g^+ = g\rho g^{-1}$. Let R be the polynomial ring in $(\prod_i d_i)^2$ variables, identified as the coordinate ring of $L(\mathcal{H}, \mathcal{H})$, and $R^{\mathbb{LU}}$ be LUIPs, i.e., the invariant ring of local unitary.

It is not feasible to apply Derksen's bound directly, as \mathbb{U} cannot be viewed as zero set of polynomials over algebraically closed field \mathbb{C} . This can be got around by considering the complexification of \mathbb{U} , which yields $\mathbb{G} = \text{GL}(d_1, \mathbb{C}) \times \cdots \times \text{GL}(d_n, \mathbb{C}) \leq \text{GL}(\mathcal{H})$. Recall that we can view R as the space of representations of \mathbb{LU} and \mathbb{G} , and note that invariant polynomials are just dimension-1 irreducible representations. Then by the correspondence between irreducible representations of \mathbb{LU} and \mathbb{G} , $R^{\mathbb{LU}} = R^{\mathbb{G}}$. Thus it is enough to get a degree bound for the action of \mathbb{G} .

To get a degree bound for the action of \mathbb{G} on R , we compute as follows. First $s := \dim(R^{\mathbb{G}}) = (\prod_i d_i)^2$. To bound $\sigma(\mathcal{H}, \mathbb{G})$, we observe that $t = \sum_i d_i^2$, $H = 1$, $d = \sum_i d_i^2$, and $A = \prod_i d_i$. Thus

$$\begin{aligned} \sigma(\mathcal{H}, \mathbb{G}) &\leq (\prod_i d_i) \sum_i d_i^2, \\ \Rightarrow \beta(\mathcal{H}, \mathbb{LU}) &= \beta(\mathcal{H}, \mathbb{G}) \leq \frac{3}{8} (\prod_i d_i) \sum_i 2d_i^2 + 2. \end{aligned}$$

In other words, two quantum states $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}$ are LU equivalent if and only if $f_i(|\Psi\rangle\langle\Psi|) = f_i(|\Phi\rangle\langle\Phi|)$ holds for any LUIPs f_i with degree less than $N(d_1, d_2, \dots, d_n)$. ■

We now provide a new view of LUIPs which leads to an algorithm for finding all of them. Let I_j be the identity operator of system \mathcal{H}_j , we can define group \mathbb{U}_i as follows,

$$\mathbb{U}_i = I_1 \otimes \cdots \otimes I_{i-1} \otimes \mathbb{U}(d_i) \otimes I_{i+1} \cdots \otimes I_n.$$

A direct but useful observation is as follows

$$\mathbb{LU} = \mathbb{U}_1 \mathbb{U}_2 \cdots \mathbb{U}_n, \text{ and } \mathbb{U}_i \subset \mathbb{LU}.$$

The advantage of this observation on studying the polynomial invariants is based on the following relation between the polynomial invariants of \mathbb{LU} , says P , and those polynomial invariants of \mathbb{U}_i s, says P_i ,

$$P = \bigcap_{i=1}^n P_i. \quad (1)$$

To see the validity of the above relation (1), we notice that $\mathbb{U}_i \subset \mathbb{LU}$ leads us to the fact that $P \subset P_i$, thus, $P \subset \bigcap_{i=1}^n P_i$. On the other hand, one can verify that for any $p \in \bigcap_{i=1}^n P_i$, $g = g_1 g_2 \cdots g_n \in \mathbb{LU}$ with $g_i \in \mathbb{U}_i$ and $|\Psi\rangle \in \mathcal{H}$, we have $p \in P$ by observing

$$\begin{aligned} &p(g|\Psi\rangle\langle\Psi|g^{-1}) \\ &= p(g_1 \cdots g_n |\Psi\rangle\langle\Psi| g_n^{-1} \cdots g_1^{-1}) \\ &= p(g_2 \cdots g_n |\Psi\rangle\langle\Psi| g_n^{-1} \cdots g_2^{-1}) \\ &= \cdots \\ &= p(|\Psi\rangle\langle\Psi|). \end{aligned}$$

Therefore, $P \supset \bigcap_{i=1}^n P_i \Rightarrow P = \bigcap_{i=1}^n P_i$.

P_i s are very easy to compute in the following way. Suppose $|\alpha\rangle = \sum_{j_1, j_2, \dots, j_n} x_{j_1 j_2, \dots, j_n} |j_1 j_2, \dots, j_n\rangle$ with variables $x_{j_1 j_2, \dots, j_n} \in \mathbb{C}$. Notice that any homogenous polynomial $q \in P_i$ is a polynomial on the elements of the

reduced density matrix $\rho_i = \text{Tr}_i |\alpha\rangle\langle\alpha| = [u_{kl}^{(i)}(\mathbf{x})]$. We know that $u_{kl}^{(i)}(\mathbf{x})$ are quadratic polynomial of \mathbf{x} . On the other hand, any polynomial on $u_{kl}^{(i)}(\mathbf{x})$ s is an element of P_i . Therefore, P_i is the ring with generators $u_{kl}^{(i)}(\mathbf{x})$. More precisely,

$$P_i = \mathbb{C}[u_{k,l}^{(i)}(\mathbf{x})].$$

Notice that P is fully characterized by the intersection of polynomial rings P_i , we can conclude that the degree of LUIP must be even. The homogeneous LUIPs with degree $2m \in \mathbb{N}$ is

$$Q_{(2m)} := \bigcap_{i=1}^n \text{span}\{[u_{kl}^{(i)}(\mathbf{x})]\}^m, \quad (2)$$

where the product of two set of polynomials is defined as $A \times B = \{a \times b : a \in A, b \in B\}$.

We know that the set of all LUIPs with fixed degree form a vector subspace over \mathbb{C} . $Q_{(2m)}$ can be viewed as a linear subspace. Therefore, Eq.(2) indicates that the subspace $Q_{(2m)}$ is the intersection of n -subspaces, which are easily computed.

In order to characterize the LU equivalence class, according to Theorem 1, we only need to find all LUIPs with degree less than $N(d_1, d_2, \dots, d_n)$. Moreover, a basis of such LUIPs is sufficient to decide the LU equivalent class instead of the set of all such LUIPs.

$$R := \{q_{(2m),j} : 0 \leq j \leq r(2m), 0 \leq 2m \leq N(d_1, d_2, \dots, d_n)\},$$

where $q_{(2m),1}, q_{(2m),2} \cdots q_{(2m),r(2m)}$ is a basis of $Q_{(2m)}$. The above equation indeed provides an algorithm to separate local unitary equivalent orbits by noticing that computing a basis of $Q_{(2m)}$ is simple after obtaining the subspace description of $Q_{(2m)}$.

On the other hand, the characterization of P can be given as

$$P = \bigcap_{i=1}^n P_i = \text{span} \bigcup_{m=1}^{\infty} Q_{(2m)} = \text{span} \bigcup_{j=1}^{\infty} R^j.$$

We have already provided a method to separate the LU orbits of pure states, interestingly, this can help us to separate the LU orbits of mixed states by noticing that two n -partite mixed states are LU equivalent if and only if their purifications, two $n+1$ -partite mixed states, are LU equivalent [22].

The LU equivalence of quantum states has the application on the equivalence between quantum channel, where two quantum channel \mathcal{E} and \mathcal{F} are said to be equivalent if there are unitary channels \mathcal{U} and \mathcal{V} such that

$$\mathcal{F} = \mathcal{V} \circ \mathcal{E} \circ \mathcal{U}.$$

Here, unitary channels \mathcal{U} and \mathcal{V} can be regarded as encoding channel and decoding channel, respectively. It is direct to verify that \mathcal{E} and \mathcal{F} have the same ability on transmit information.

One can observe that $\mathcal{E}, \mathcal{F} : L(\mathcal{H}_A) \rightarrow L(\mathcal{H}_B)$ are equivalent if and only if their choi-matrices are LU equivalent, where the choi-matrix of quantum channel $\mathcal{E}(\cdot) = \sum_i E_i \cdot E_i^\dagger$ is defined as bipartite mixed state (non-normalized) $\rho_{AA'} = (\mathcal{I}_{A'} \otimes \mathcal{E})(|\Psi\rangle\langle\Psi|)$ with $|\Psi\rangle = \sum_{j=1}^d |i\rangle|i\rangle$ and noiseless channel $\mathcal{I}_{A'}$ on quantum system $\mathcal{H}_{A'}$ which has the same dimension d with system \mathcal{H}_A . Invoking our previous argument, this problem can be reduced to the LU equivalence between tripartite pure states, which can be solved using our algorithm on searching the LUIPs of this tripartite system.

In the following, we demonstrate that the concept of LUIPs is a powerful tool on study the LU equivalence between quantum states by showing the following result,

Theorem 2. *There is some $r \in \mathbb{N}$ such that $|\Psi\rangle^{\otimes r}, |\Phi\rangle^{\otimes r}$ are LU equivalent, then $|\Psi\rangle, |\Phi\rangle$ are LU equivalent.*

Remark:— The bipartite case is obvious since the structure of bipartite pure entanglement is simply determined by it Schmidt coefficients. *Proof:*—By applying the above procedure to study the LUIPs for r copies of states, says $|\Psi\rangle^{\otimes r}, f_1 f_2 \cdots f_r$ is an LUIP for any LUIPs for the original system f_1, f_2, \dots, f_k of degree l_1, l_2, \dots, l_k with $\sum l_k$ divisible by $2r$. More precisely, we can conclude $\prod_{i=1}^k f_i(|\Psi\rangle\langle\Psi|) = \prod_{i=1}^k f_i(|\Phi\rangle\langle\Phi|)$.

We choose the degree 2 LUIP f_0 be the square of 2-norm function, which satisfies that $f_0(|\Psi\rangle\langle\Psi|) = f_0(|\Phi\rangle\langle\Phi|)$. Then according to the LU equivalence of $|\Psi\rangle^{\otimes r}$ and $|\Phi\rangle^{\otimes r}$, we know the following equation is valid for any LUIP f' of degree $2l$ with $i+l$ divisible by r ,

$$f_0^i(|\Psi\rangle\langle\Psi|)f'(|\Psi\rangle\langle\Psi|) = f_0^i(|\Phi\rangle\langle\Phi|)f'(|\Phi\rangle\langle\Phi|)$$

Therefore, $f'(|\Psi\rangle\langle\Psi|) = f'(|\Phi\rangle\langle\Phi|)$ is valid for any LUIP g . By invoking the result that LUIPs are sufficient to separate any two distinct orbits under local unitary, we can conclude that $|\Psi\rangle, |\Phi\rangle$ are LU equivalent. ■

It is direct to see that similar statement is true for mixed states, and for quantum channels by recalling our previous arguments.

To see the power of our method on constructing LUIPs, we apply it on studying the SLOCC equivalence, where the pure state case was recently solved by Gour and Walach using Schur-Weyl duality [13].

We apply Derksen's bound to the following action related to the SLOCC equivalence of pure state and obtain an explicit upper bound for the degree of some generating set of SLIPs.

Let $\mathbb{G} = \text{GL}(d_1, \mathbb{C}) \times \cdots \times \text{GL}(d_n, \mathbb{C})$ acts on \mathcal{H} in the natural way. Let R be a polynomial ring over \mathbb{C} in $\prod_i d_i$ variables, identified as the coordinate ring of \mathcal{H} , and $R^{\mathbb{G}}$ denote the invariant ring.

Note that G is the zero locus of $\det(z_{i,j}^{(k)})_{i,j \in [d_k]}$ for $k \leq n$, we see that $t = \sum_i d_i^2$, $H = \max\{d_i \mid i \leq n\}$, and $d := \dim(\mathbb{G}) = t - n$, $A = n$, and get

$$\sigma(\mathcal{H}, \mathbb{G}) \leq \max(d_i)^n \cdot n^{\sum_i d_i^2 - n}.$$

As $s := \dim(R^{\mathbb{G}}) \leq \prod_i d_i$, we get

$$\beta(\mathcal{H}, \mathbb{G}) \leq \frac{3}{8} \prod_i d_i \max(d_i)^{2n} \cdot n^{\sum_i 2d_i^2 - 2n}.$$

In [13], and algorithm to construct the SLIPs for fixed degree is given, here we provide an alternative algorithm.

According to the argument of the local unitary case, we only need to compute the invariant polynomials of group SL_i , with $\text{SL}_i = I_1 \otimes \cdots \otimes I_{i-1} \otimes \text{SL}(d_i) \otimes I_{i+1} \cdots \otimes I_n$ and $\text{SL}(d_i)$ standing for $d_i \times d_i$ matrix with determinant 1. Therefore, we can regard the multipartite state as bipartite pure state which is just a matrix, says X , and the action of the group is just the left matrix multiplication, i.e., $X \rightarrow LX$ with $\det(L) = 1$. Fortunately, the invariant polynomials of such map are fully characterized by the determinant of all square matrix with columns catching from X . Therefore, these argument simple implies an algorithm whose output is the SLIPs for the multipartite system \mathcal{H} .

Using similar proof technique as that of Theorem 2, we can know that the statement is also true for stable pure states under SLOCC by employing the result from [13] that the orbits of stable states can be distinct by SLIPs.

Discussion—In order to study the equivalence relation of mixed states under the acting group $\text{SL} = \text{SL}(d_1) \otimes \text{SL}(d_2) \otimes \cdots \otimes \text{SL}(d_n)$, i.e., two mixed states ρ and σ are said equivalent under rank one SLOCC if there is some $A = A_1 \otimes \cdots \otimes A_n$ such that ρ is proportional to $A\sigma A^\dagger$, $\{A_j\}_{j=1}^n$ are invertible $m_j \times m_j$ matrices. One can derive the following observation,

Proposition 1. *ρ and σ are equivalent under rank one SLOCC if and only if there is some $s \in \text{SLU}$ such that $|\Psi\rangle$ is proportional to $s|\Phi\rangle$, where $|\Psi\rangle, |\Phi\rangle \in \mathcal{H} \otimes \mathcal{H}_{n+1}$ are some purification of ρ, σ with d_{n+1} being the dimension of \mathcal{H}_{n+1} , and*

$$\text{SLU} \equiv \text{SL}(d_1) \otimes \text{SL}(d_2) \otimes \cdots \otimes \text{SL}(d_n) \otimes \text{U}(d_{n+1}).$$

It is showed that SLIPs can be used to determine whether two stable states in \mathcal{H} belong to the same SLOCC class, where a state $|\Psi\rangle \in \mathcal{H}$ is said to be *stable* if its orbit $S|\Psi\rangle$ is closed [13]. Proposition 1 motivates us to generalized the definition of stable pure state [13] to mixed state case. The following definition coincides with that of [13] for pure states.

Definition 1. *Mixed state ρ of system \mathcal{H} with some purification $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}_{n+1}$ is said to be stable under SLOCC if its orbit closed, where the orbit of ρ under SLU is defined as $\{A \otimes u|\Psi\rangle, A \in \mathbb{S}, u \in \text{U}(d_{n+1})\}$ with d_{n+1} being the dimension of \mathcal{H}_{n+1} .*

In order to study the SLOCC equivalence of mixed state, we only need to classify the distinct orbits of pure states under group SLU . Similarly, we employ the concept of invariant polynomial under group SLU .

An SLU -invariant polynomial (SLUIP) is a polynomial $f : \mathcal{H}_n \rightarrow \mathbb{C}$ that satisfies

$$f(s|\Psi\rangle\langle\Psi|s^+) = f(|\Psi\rangle\langle\Psi|) \quad \forall s \in \mathbb{S} \text{ and } \forall |\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}_{n+1},$$

Our previous method can be used to construct the whole set of SLUIPs. Now, it is desirable to know that whether SLU -invariant polynomial is finitely generated. Moreover, we have the following conjecture,

Conjecture 1. *There is a finite set of generators of invariant polynomial under group SLU , where each generator has degree less than $M(d_1, d_2, \dots, d_{n+1})$.*

If the above conjecture is valid, we are able to generalize the result of [13], we show that the invariant polynomial of group SLU can be used to determine whether two stable mixed states in \mathcal{H} belong to the same orbit. More precisely, Let $|\Psi\rangle, |\Phi\rangle \in \mathcal{H} \otimes \mathcal{H}_{n+1}$ be two stable states (i.e. states $|\Psi\rangle$ whose orbits $SLU|\Psi\rangle$ are closed). Then, they share the same orbit if and only if for all homogeneous invariant polynomial of group SLU of degree k , f_k, g_k , with $h_k(|\Psi\rangle) \neq 0$,

$$f_k(|\Psi\rangle\langle\Psi|)/h_k(|\Psi\rangle\langle\Psi|) = f_k(|\Phi\rangle\langle\Phi|)/h_k(|\Phi\rangle\langle\Phi|).$$

Conclusion—In this paper, we give a characterization of multipartite entanglement by exploiting a systematic method to describe the ring of all LUIPs. More precisely, we then provide an algorithm to construct a set of generators of the ring of LUIPs. By using the concept of LUIPs, we are able to show that multipartite entanglement is additive in the sense that two multipartite states are LU equivalent if and only if n -copies of these two states are LU equivalent for some n . Interestingly, this idea can be used to study the multipartite mixed entanglement in terms of equivalence classes of states under SLOCC. We construct the whole set of SL-invariant polynomials (SLIPs) for the pure state case. Then the problem of the equivalence between pure states under the SLU group is studied, and the set of invariant polynomials are demonstrated.

There are still several unsolved problems left in our paper. First, it is interesting to understand the computational complexity of studying the LU(or SLOCC) equivalence of given quantum states, lower bound and upper bound, especially, does polynomial degree bound of invariant polynomials exist, for the group LU(or SLOCC)? The second problem is to provide an explicit upper bound of the group of SLU .

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- [26] Let W be an algebraic set defined as the set of the common zeros of an ideal I in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , and let $A = R/I$ be the algebra of the polynomials over W . Then the dimen-

sion of W is: The maximal length d of the chains $V_0 \subset V_1 \subset \dots \subset V_d$ of distinct nonempty subvarieties.